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A DEFENSE ALLOCATION PROBLEM WITH DEVELOPMENT COSTS

By
Kenneth D. Shere
Edgar A. Cohen, Jr.

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NAVAL ORDNANCE LABORATORY, WHITE OAK, SILVER SPRING, MARYLAND

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Kenneth D. Shere
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A Defense Allocation Problem with Development Costs

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ROBERT ENNIS
Captain, USN
Commander

E. K. RITTER
By direction

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I. INTRODUCTION

Strategic system decisions are subject to various levels of review. Technical, fiscal, and political factors are all introduced during the review process. Such problems as individual system performance and mutual support and protection between proposed and existing systems are also considered. Quantitative analysis of system mixes to support and justify selection decisions is beneficial, if not required. Such analysis should include the performance of individual systems taken individually, the performance of system mixes, costs to the enemy in defeating the various mixes, and protective interactions among the members of system mixes. In this paper a method for allocating resources among strategic weapon systems for deterrence purposes is presented.

In the following section, the mathematical model is presented. This model improves upon earlier models [1-3] by accounting for the price of admission. The price of admission includes development costs and we shall sometimes use these terms interchangeably; however, it specifically *excludes* research funds. In fact, before any system is considered for development it is assumed that preliminary research has been performed.

In Sections III and IV, respectively, a method of solution is presented for an arbitrary mix of *percentage vulnerable* (PV) systems and for a general mix of PV and *numerically vulnerable* (NV) systems. The theory of max-min is extended in Section III as necessary to solve this particular problem. The notation is summarized in the Appendix.

II. MATHEMATICAL MODEL INCLUDING DEVELOPMENT COSTS

There are many ways in which system mix analyses can be conducted. In all of them the problem of characterizing the systems to be studied is of extreme importance. An intuitively appealing characterization of system alternatives results in a simple but highly versatile model. In this interpretation, due to Dr. Thomas E. Phipps [1], defender retaliatory system candidates consist of two exclusive classes.

One class comprises systems which are difficult to locate, but relatively easy to destroy once located. An example of such a mobile system is POLARIS. This type of weapon system is called *percentage vulnerable* because, for a fixed search effort by the attacker, a fixed percentage of the retaliatory weapons comes under attack. The other class consists of weapons that are easily located but difficult to destroy. MINUTEMAN is an example of such a system. These systems are called *numerically vulnerable* because the attacker's effort must be distributed among all of the weapons of the system. We assume that each retaliatory weapon system can be attacked by only one weapon system of the y-player.

The retaliator allocates x_k resources, e.g. in billions of dollars, to the kth weapon system, which costs q_k to develop and $c_k n_k$ to procure; n_k is the number of weapons in the system and c_k is the procurement cost per weapon. The number of weapons in the kth system is

$$n_k = (x_k - q_k)/c_k.$$

If w_k is the throw-weight in megatons of a weapon in the kth system, the total throw-weight for the kth system is

$$w_k n_k = (w_k/c_k)(x_k - q_k).$$

Using the theory of random search [cf. 4], it can be shown [cf. 1,2] that the fraction of weapons destroyed in the ith PV system is given by $1-\exp[-a_i(y_i-r_i)]$; y_i is the amount of resources the attacker allocates to blunting, on first strike, the corresponding retaliator's system and r_i is the attacker's development cost. a_i is the "vulnerability" of the ith system measured, e.g. in inverse billions of dollars. Consequently, the residual value of the ith percentage vulnerable system is

(2.1) $v_i(x_i-q_i)exp[-a_i(y_i-r_i)]$,

where $v_i \equiv w_i/c_i$.

For an NV system, the attacker's resources must be distributed among all the weapons of the system. Therefore, the survival probability is $exp[-a_j(y_j-r_j)/(x_j-q_j)]$ and the residual value is

(2.2) $v_j(x_j-q_j)exp[-a_j(y_j-r_j)/(x_j-q_j)]$

Throughout this article the subscript i will be used for percentage vulnerable systems and the subscript j will be used for numerically vulnerable systems.

In deriving (2.1) and (2.2) it has been implicitly assumed that $x_k > q_k$ and $y_k > r_k$. Combining (2.1) and (2.2) and applying physical reasoning when the implicit assumptions are violated, we obtain the residual value for the retaliator's system mix:

$$(2.3a)\quad F(x,y) = \sum_{i=1}^{n} f_i(x_i,y_i) + \sum_{j=n+1}^{m} f_j(x_j,y_j),$$

where

$$(2.3b)\quad f_i(x_i,y_i) \equiv \begin{cases} 0 & : \ x_i \le q_i \\ v_i(x_i-q_i)exp[-a_i(y_i-r_i)] & : \ x_i > q_i \ , \ y_i > r_i \\ v_i(x_i-q_i) & : \ x_i > q_i \ , \ y_i \le r_i \end{cases}$$

and

$$(2.3c)\quad f_j(x_j,y_j) \equiv \begin{cases} 0 & : \ x_j \le q_j \\ v_j(x_j-q_j)exp[-a_j(y_j-r_j)/(x_j-q_j)] & : \ x_j > q_j \ , \ y_j > r_j \\ v_j(x_j-q_j) & : \ x_j > q_j \ , \ y_j \le r_j . \end{cases}$$

The attacking y-player, having full knowledge of the retaliator's allocation, allocates his funds to minimize the x-player's retaliatory capability, F(x,y). Consequently, the retaliator allocates his funds in a manner which maximizes

this minimum; i.e., the objective is to determine the optimal strategies u for the retaliator and w for the attacker so that

(2.4) $$V \equiv F(u,w) = \mathrm{Max}_x[\mathrm{Min}_y F(x,y)] \equiv \mathrm{Max}_x\ P(x) = P(u).$$

It is also desirable to know V. An unusually low value would indicate that a large infusion of funds by the defender is necessary.

Defining X and Y to be the retaliator's and attacker's total resources, respectively,

(2.5) $$\sum_{i=1}^{n} x_i + \sum_{j=n+1}^{m} x_j = X$$

and

(2.6) $$\sum_{i=1}^{n} y_i + \sum_{j=n+1}^{m} y_j = Y$$

Of course, $x_k \geq 0$, $y_k \geq 0$, $a_k > 0$, $v_k > 0$, $r_k > 0$ and $q_k > 0$ for each weapon system. The limitations of this model and the procedures for determining the parameters a_k and v_k have been discussed by Phipps [2].

The mathematical model (2.3)-(2.6) with no development costs (i.e., $r_k = q_k = 0$ for all k) has been completely solved by Danskin [3]. The inclusion of development costs is the first of several improvements in the existing theory necessary for a realistic model.

III. PV SYSTEMS

In this section the finite allocation problem with development costs is resolved for an arbitrary mix of PV systems, (2.3)-(2.6) with m=n. This problem is summarized by:

(3.1) *Given:* $$F(x,y) = \sum_{i=1}^{n} f_i(x_i,y_i)$$

(3.2) *Constrained by:* $\sum_{i=1}^{n} x_i = X$; $\sum_{i=1}^{n} y_i = Y$

$x_i \geq 0$; $y_i \geq 0$

(3.3) *Determine:* u, w and V where $V = F(u,w) \equiv \text{Max}_x P(x) \equiv \text{Max}_x[\text{Min}_y F(x,y)]$

We hypothesize that $X > \text{Max } q_k$ and $Y > \text{Max } r_k$. Physically this means that the retaliator considers only those systems for which he can afford to procure at least one weapon and that the attacker can afford at least a limited counter to any retaliatory system. Mathematically, these assumptions assure a positive residual value.

Let $F(u,w) = \text{Max}_x \text{Min}_y F(x,y)$. It is shown that (u,w) is the solution of a game defined on suitable subsets of the x-space and y-space. It is also shown that either $u_i = 0$ or $u_i > q_i$, and a constructive procedure for solving (3.1)-(3.3) is then presented.

Lemma 3.1. If $P(x) = F(x,\eta(x)) > 0$, then $x_k > q_k$ and $\eta_k > r_k$ for some k.

Proof. Define $\Gamma \equiv \{i : x_i > q_i\}$ and let us suppose that $\eta_i \leq r_i$ for all $i \in \Gamma$. Then $P(x) = \sum_\Gamma v_i(x_i - q_i)$. Select $k \in \Gamma$ and define η^* by $\eta_i^* = 0$ $(i \neq k)$ and $\eta_k^* = Y$. Since $Y > r_k$, $P(x) > F(x,\eta^*)$ contrary to the definition of P(x). Consequently, there exists a $k \in \Gamma$ such that $\eta_k > r_k$.

We note that $V = P(u) > 0$ because $P(x) > 0$ for $x = (X,0,0,\ldots,0)$.

Lemma 3.2. If $P(x) = F(x,\eta(x)) > 0$, $x_i \leq q_i$ implies $\eta_i = 0$.

Proof. Suppose, to the contrary, that $x_k \leq q_k$ and $\eta_k > 0$ for some $i = k$. Define $\Gamma \equiv \{i : 0 \leq x_i \leq q_i\}$ and $\sigma \equiv \sum_\Gamma \eta_i$. From Lemma 3.1, $x_i > q_i$ and $\eta_i > r_i$ for some $i = k$. Define η^* by:

$$\eta_i^* \equiv \begin{cases} 0 & : i \in \Gamma; \\ \eta_i & : i \in \Gamma, \ i \neq k; \\ \eta_i + \sigma & : i = k. \end{cases}$$

$F(x,\eta^*) < F(x,\eta)$ contrary to the definition of P(x) and η.

The following two lemmas are modifications of Gibbs' Lemma [cf. 3, p. 10]. The first modification is trivial.

Lemma 3.3. Let $f_i(x_i)$ be differentiable. Let $z = (z_1,\ldots,z_n)$ maximize $\Sigma_i f_i(x_i)$ constrained by

$$\Sigma_i\, x_i = X > 0 \;;\; x_i \geq q_i \geq 0.$$

Then there exists a number λ so that

$$\begin{aligned} f_i'(z_i) &= \lambda \quad : \quad z_i > q_i; \\ &\leq \lambda \quad : \quad z_i = q_i. \end{aligned}$$

Lemma 3.4 (Modified Gibbs' Lemma). Let $f_i(x_i)$ be continuous with right- and left-derivatives. Let $z = (z_1,\ldots,z_n)$ maximize $\Sigma_i f_i(x_i)$ constrained by

$$\Sigma_i x_i = X \;,\; x_i \geq 0$$

and $X > 0$. Then there exists a number λ such that

$$f_i'(z_i^+) \leq \lambda \quad \text{for all } i;$$

$$f_i'(z_i^-) \geq \lambda \quad \text{for all } i,\ z_i > 0.$$

Furthermore, if $f_i(x_i)$ is differentiable at $x_i = z_i > 0$, for some i, λ is unique.

Proof. Suppose $z_i > 0$ and $0 \leq \varepsilon \leq z_i$; define

$$F(\varepsilon) \equiv f_i(z_i-\varepsilon) + f_j(z_j+\varepsilon) + \sum_{k\neq i,j} f_k(z_k)$$

The altered set z still satisfies the constraints. Therefore, $F(\varepsilon)$ is maximal at $\varepsilon = 0$ and $F'(0) \leq 0$, i.e.

$$f_j'(z_j^+) \leq f_i'(z_i^-)$$

for all j. We now choose any λ such that

$$\text{Max}_j\ f_j'(z_j^+) \leq \lambda \leq \underset{z_i>0}{\text{Min}}\ f_i'(z_i^-).$$

We note that whenever $f_i(x_i)$ is differentiable at $z_i > 0$, for some i, the choice of λ is uniquely determined by $\lambda = f_i'(x_i)$.

The following two lemmas demonstrate that paying a portion of development costs without procuring any weapons is a waste of resources. Although these lemmas seem physically obvious, they are not mathematically obvious.

Lemma 3.5. Either $w_i = 0$ or $w_i > r_i$.

Proof. Define $A' \equiv \{i : 0 < w_i \leq r_i\}$. By Lemma 3.1, there exists $i = k$ such that $u_k > q_k$ and $w_k > r_k$. Select $i = k' \in A'$ and define w^* as the n-dimensional vector:

$$w_i^* = \begin{cases} 0 & : \; i = k' \\ w_i & : \; i \neq k,k' \\ w_i + w_{k'} & : \; i = k \end{cases}$$

It follows that $F(u,w) > F(u,w^*)$ contrary to the definition of $\text{Max}_x \text{Min}_y$. Hence A' is empty.

Lemma 3.6. Either $u_i = 0$ or $u_i > q_i$

Proof. Define $B \equiv \{i : u_i > q_i\}$ and $A \equiv \{i : w_i > r_i\}$. From Lemma 3.2, $A \subseteq B$. Then (3.1) becomes, at x = u,

$$F(u,y) = \Sigma_B \, f_i(u_i,y_i). \tag{3.4}$$

Application of the Modified Gibbs' Lemma to (3.4) yields

$$a_i v_i (u_i - q_i) \, exp[-a_i(w_i - r_i)] \begin{array}{l} = \mu \; : \; w_i > r_i; \\ 0 \leq \mu \; : \; w_i = 0 \end{array} \tag{3.5}$$

For each $i \in A$,

$$w_i = r_i + (1/a_i) \, \ell og[a_i v_i (u_i - q_i)/\mu]; \tag{3.6}$$

otherwise, $w_i = 0$. Substitution of (3.6) into the y-constraint of (3.2) yields

$$\Sigma_A \{r_i + (1/a_i) \, \ell og[a_i v_i (u_i - q_i)/\mu]\} = Y. \tag{3.7}$$

For each possible A, (3.7) either has a unique solution for μ or no solution. Unfortunately there is no *a priori* way of obtaining A, at this point of the analysis, even if u were known. The residual value of the retaliator's mix is:

$$(3.8) \qquad V = \Sigma_A \mu/a_i + \Sigma_{B-A} v_i(u_i-q_i).$$

Thus A is determined by finding $\mu(A)$ for each possible choice of A and selecting the choice which minimizes V. We note that $a_i v_i(u_i-q_i) > \mu$ for each $i \in A$, but the converse is not necessarily valid.

Assume that $0 < u_k < q_k$ for some k. We note that $k \notin B$ and by Lemma 3.2, $A \subsetneq B$. Select $j \in B$ and define u^* for suitable $\varepsilon > 0$ by:

$$u_i^* = u_i : i \neq j,k;$$
$$u_j^* = u_j + \varepsilon;$$
$$u_k^* = u_k - \varepsilon.$$

We can choose ε so small that the possible choices for A are unaffected. Again applying the modified Gibbs' lemma, (3.6)-(3.8) hold with u, w, μ and V replaced by u^*, w^*, μ^* and V^*. For each possible choice A one of the following statements is true:

$$\mu = \mu^* \text{ and } \Sigma_{B-A} v_i(u_i-q_i) < \Sigma_{B-A} v_i(u_i^*-q_i) : j \notin A;$$

$$\mu < \mu^* \text{ and } \Sigma_{B-A} v_i(u_i-q_i) = \Sigma_{B-A} v_i(u_i^*-q_i) : j \in A.$$

Thus $V^* > V$, and $x = u$ could not have been the retaliator's optimal allocation. The lemma is established.

Lemma 3.7. $P(x) \equiv \text{Min}_y F(x,y)$ is a continuous function of x.

Theorem 3.8. If the choice of A is unique (for a given u) and (u,w) solves (3.1)-(3.3), then (u,w) is also a solution of the game:

(3.9) *Given:* $F(x,y) = \Sigma_A v_i(x_i-q_i)exp[-a_i(y_i-r_i)]$

$$+ \Sigma_{B-A}\ v_i(x_i-q_i)$$

(3.10) *Constrained by:* $\Sigma_B\ x_i = X\ ,\ \Sigma_A y_i = Y$

$$x_i \geq q_i (i \in B)\ ,\ y_i \geq r_i\ (i \in A)$$

(3.11) *Determine:* $V = Max_\beta P_A(x) = Max_\beta[Min_\alpha F(x,y)]$

$$= Min_\alpha[Max_\beta F(x,y)] \equiv V_{B,A}$$

where

$\beta(B) \equiv \{x : x_i = 0$ *for* $i \notin B$ *and (3.10) satisfied*$\}$;

$\alpha(A) \equiv \{y : y_i = 0$ *for* $i \notin A$ *and (3.10) satisfied*$\}$

Proof. Since A is unique and $P(x) = Min\{P_{A'}(x) : A' \subseteq B\}$ is continuous at $x = u$, A is the minimizing set for each point x in some ε-neighborhood of $x = u$, $E \subseteq \beta$; that is, $P_A(x) = P(x)$. Hence $P_A(u) \geq P_A(x)$ for each $x \in E$. Since A is unchanged for each $x \in E$, $y(x) \in \alpha$ and $F(x,y)$ is given by (3.9). From the concave-convex behavior of $F(x,y)$, (u,w) is a saddle-point. Since any saddle-point of (3.9) satisfying (3.10) is a solution to the game (3.9)-(3.11), the theorem is established.

Of course, (3.1)-(3.3) is not yet resolved because there is no *a priori* scheme for determining A,B and the uniqueness of A.

The uniqueness of A does not in itself present a real problem. Because of the political nature of this subject, none of the parameters are precisely known. For example, it is difficult for a retaliator to estimate his own resources or budget over a five to ten year period; his estimate of the attacker's budget is even more tenuous. Consequently, a parameter analysis must be performed for any practical application to strategic systems. Except for isolated points the choice of A is generally unique. The isolated points may then be determined by continuity.

Under these circumstances the attacker's choice is based partially on internal politics.

For the remainder of this section, A is assumed unique for some optimal allocation $x = u$. Under this hypothesis it is shown below that $A = B$ and an algorithm for determining B is presented.

Application of Gibbs' Lemma 3.3 to the game (3.9)-(3.11), which has an interior solution, yields for $i \in B$:

(3.12a) $v_i exp[-a_i(w_i-r_i)] = \lambda_B \; : \; i \in A;$

(3.12b) $v_i \leq \lambda_B \; : \; i \notin A.$

Upon substitution of (3.12) into the y-constraint of (3.10), we determine λ from

(3.13) $\Sigma_A[r_i+(1/a_i)\ell og v_i/\lambda_B] = Y.$

The residual value of (3.9)-(3.11) is

(3.14) $V_{B,A} = \lambda_B(X-\Sigma_A q_i-\Sigma_{B-A}q_i).$

Lemma 3.9. Let C be any set such that $B \supseteq C \supseteq A$ and let $V_{C,A}$ be the value of the game determined by (3.9)-(3.11) with B replaced by C. Then there is a number $\lambda_C \geq \lambda_B$ such that

(3.15) $V_{C,A} = \lambda_C[X-\Sigma_C q_i].$

Proof. By applying Lemma 3.3 to (3.9)-(3.11) with B replaced by C, one finds for the optimal x,y and for $i \in C$:

(3.16a) $v_i = \lambda_C \; : \; x_i > q_i \; , \; y_i = 0;$

(3.16b) $v_i exp[-a_i(y_i-r_i)] = \lambda_C \; : \; x_i > q_i \; , \; y_i > 0;$

(3.16c) $v_i \leq \lambda_C \; : \; x_i = q_i \; , \; y_i = 0;$

(3.16d) $v_i exp[-a_i(y_i-r_i)] \leq \lambda_C \; : \; x_i = q_i \; , \; y_i > 0.$

Hence

(3.17) $\Sigma_A[r_i+(1/a_i)\log(v_i/\lambda_C)] \leq \Sigma_A y_i = Y.$

A comparison of (3.17) to (3.13) shows that $\lambda_C \geq \lambda_B$. From (3.16) the value is

$$V_{C,A} = \Sigma_{C-A} v_i(x_i-q_i) + \Sigma_A v_i(x_i-q_i)\exp[-a_i(y_i-r_i)]$$
$$= \lambda_C[X-\Sigma_C q_i].$$

Theorem 3.10. A = B

Proof. Suppose to the contrary that $A \subset B$. Set C = A in Lemma 3.9 and compare (3.15) to .... It is seen that $V_{A,A} > V_{B,A} = V$. Select $x' \in \beta(A)$ such that $P_A(x') = V_{A,A}$. By Lemma 3.2,

$$P(x') = P_A(x') = V_{A,A} > V.$$

This contradicts the definition of V as $\mathrm{Max}_x P(x)$. Hence B = A.

This theorem shows that the retaliator should not invest in a new system unless it is of sufficient value for the attacker to pay the penalty for at least a limited counter.

Corollary 3.11. Let $\mathcal{B} \equiv \{\Gamma \subseteq \{1,2,\ldots,n\}$: *(3.13) has a solution with* $A = \Gamma\}$. Then $V = \mathrm{Max}_{\mathcal{B}} V_{\Gamma,\Gamma}$.

Proof. By Theorems 3.8 and 3.10, $V = V_{A^*,A^*}$ for some $A^* \in \mathcal{B}$. Suppose that

$$V_{A,A} \equiv \mathrm{Max}_{\mathcal{B}}\, V_{\Gamma,\Gamma} > V.$$

Select $x_A \in \beta(A)$ such that $P_A(x_A) = V_{A,A}$. From Lemma 3.2, $P(x_A) = P_A(x_A) > V$. This contradicts the definition of V.

Theorem 3.10 shows that A = C = B in Lemma 3.9. It may be shown for any set D that

$$V_{D,D} = \lambda_D\,[X-\Sigma_D q_i]$$

where (3.16) holds with "C" replaced by "D." Consequently $\lambda_D \geq \lambda_D^*$, where λ_D^* is the solution of

$$(3.13)' \quad \Sigma_D[r_i+(1/a_i)\log v_i/\lambda_D^*] = Y.$$

If $D = B$, $\lambda_D = \lambda^*_D$.

A procedure for solving (3.1)-(3.3) can be specified.

Algorithm 3.12

1. Select $D \subseteq \{1,2,\ldots,n\}$.
2. Solve (3.13)' for λ_D^* and evaluate
   $$V_D^* = \lambda_D^*(X-\Sigma_D q_i)$$
3. Go to step #1 until all possible choices of D are exhausted.
4. Determine B from $V_{B,B} = \text{Max } V_D = \text{Max } V_D^*$.
5. Solve for $w_i (i \in B)$ from (3.12a) and set $w_i = 0 \; (i \notin B)$.
6. Determine μ from (3.8), i.e., $\mu = V/\Sigma_B(1/a_i)$.
7. Determine $u_i (i \in B)$ from (3.5).
8. Set $u_i = 0 \; (i \notin B)$.

Example. As an example we set $n = 2$, $r_1 = r_2 = 1/10$, $q_1 = 1/10$, $q_2 = 2/10$, $v_1 = 1$, $v_2 = 2/3$, $a_1 = 1$, $a_2 = 1/2$ and $X = Y = 1$. There are three possible choices for D, {1}, {2} and {1,2}. Considering these choices in turn (steps 1-3), we obtain Table 3.1

D	λ	V^*
1	0.40657	0.36591
2	0.425085	0.34007
1,2	0.58380	0.40866

Table 3.1. Determination of Residual Value

Continuing to follow Algorithm 3.12 one finds:

$B = \{1,2\}$

$w = (0.637, 0.363)$

$\mu = V/3 = 0.13622$

$u = (1/3, 2/3)$

No procedure for obtaining the solution in closed form appears to be available. A few qualitative results, as

$$Y > \Sigma_B[r_i + (1/a_i)\log(v_i/v_{Min})]$$

where $v_{Min} \equiv Min_B v_i$, can be easily deduced and may help limit the number of cases that must be considered.

IV. GENERAL MIX OF PV AND NV SYSTEMS

In this section we first demonstrate that additional resources should be invested in at most one NV system. A method of solution for a general mix of PV and NV systems is then presented.

Lemmas 3.1-3.6 also apply to NV systems, as can be demonstrated with only trivial modifications in their proofs. As an equivalent to Theorem 3.8,

Theorem 4.1. If the choice of A is unique and (u,w) solves (2.3)-(2.6) with $n = 0$, then (u,w) also is a solution of:

(4.1) *Given:* $F(x,y) = \Sigma_A\, v_i(x_i-q_i)\exp[-a_i(y_i-r_i)/(x_i-q_i)]$

$$+ \Sigma_{B-A}\, v_i(x_i-q_i)$$

(4.2) *Constrained by:* $\Sigma_B\, x_i = X$, $\Sigma_A\, y_i = Y$

$$x_i \geq q_i \; , \; y_i \geq r_i$$

(4.3) *Determine:* $V = Max_\beta P_A(x) = Max_\beta Min_\alpha F(x,y)$.

The proof is similar to the proof of Theorem 3.7. It is seen that $P_A(x) = P(x)$ in some neighborhood of $x = u$ and since $P_A(x)$ is a convex function, $P_A(u)$ is the maximum of $P_A(x)$ in α. We further note that $P_A(x)$ is a convex function on a convex set; hence, its maximum must occur at an extreme, or corner, point. We have, therefore, established a basic result.

Theorem 4.2. At most one numerically vulnerable system should be developed.

Applying a trivial modification of Gibbs' lemma to (4.1)-(4.3) yields

$$(4.5) \qquad a_j v_j \exp[-a_j(y_j - r_j)/(x_j - q_j)] = \mu : y_j > r_j \ , \ j \in B$$

We note that B has precisely one element, j. Applying (4.2) we determine μ from

$$r_j + [(x - q_j)/a_j] \log a_j v_j/\mu = Y.$$

By trying all possible combinations, B is selected so that P(x) is maximized.

Finally we consider the general mix of PV and NV systems given by (2.3)-(2.6). As in the numerically vulnerable case it is determined [cf. 3, Theorem II, p. 64] that investment in NV systems should be limited to at most one weapon system. The amount of this investment is considered a parameter and we, thereby, reduce the problem to the PV problem considered in Section IV. Unfortunately, we have not yet determined an elegant way of finding the best value of the parameter; however, a value can be obtained by a computer search.

V. CONCLUSIONS

The problem of allocating resources to a general mix of percentage vulnerable and numerically vulnerable systems has been resolved. The price of admission has been included in the mathematical model. It has been shown that at most one numerically vulnerable system should receive additional resources. It has also

been shown that a system should not be developed and procured by the retaliator unless it is of sufficient value to force the attacker to invest resources to develop and procure a counter.

The model is limited by the assumption of a one-to-one correspondence between attacking systems and retaliating systems. This means, for example, that none of the attacker's systems can attack two of the retaliator's systems. Another limitation occurs in the classification itself. It would be desirable to investigate the effects of including a system which is intermediate between PV and NV. The extension of this model to include operating costs and time phasing of purchases is of major importance. Also of major importance is the extension of this model to include the value of committed resources. In [5], committed resources are included, but development costs are excluded.

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APPENDIX

Notation

(a_k)	vulnerability of kth system
A	$\{i : w_i > r_i\}$
B	$\{i : u_i > q_i\}$
$\mathcal{B}$	$\{\Gamma \subseteq \{1,2,\ldots,n\}$: (3.13) has a solution with A replaced by $\Gamma\}$
$f_k(x_k,y_k)$	residual value of retaliator's kth system (2.3)
$F(x,y)$	residual value of retaliator's system mix
i	refers to PV system
j	refers to NV system
m	total number of retaliator's (or attacker's) systems
n	number of PV systems
$P(x)$	security function, $\mathrm{Min}_y\ F(x,y)$
$P_A(x)$	$\mathrm{Min}_{\alpha(A)}\ F(x,y)$
q_k	retaliator's price of admission for kth system
r_k	attacker's price of admission for kth system
$u=(u_k)$	retaliator's optimal allocation
$v=(v_k)$	value (of the retaliator's kth system)
v_{Min}	$\mathrm{Min}_B\ v_i$
V	residual value of retaliator's system mix, $\mathrm{Max}_x\ P(x)$
$V_{B,A}$	value of game (3.9)-(3.11)
V_D^*	$\lambda_D^*(X-\Sigma_D q_i)$
$w=(w_k)$	attacker's optimal allocation
$x=(x_k)$	retaliator's allocation
X	retaliator's total resources
$y=(y_k)$	attacker's allocation
Y	attacker's total resources

Notation (Continued)

$\alpha(\Gamma)$	$\{y : y_i = 0$ *for* $i \notin \Gamma$ *and* (3.10) *is satisfied with* $A=\Gamma\}$
$\beta(\Gamma)$	$\{x : x_i = 0$ *for* $i \notin \Gamma$ *and* (3.10) *is satisfied with* $B=\Gamma\}$
$\eta(x)$	optimal attacker's allocation for a given retaliator's allocation
λ, λ_B	constant of Gibbs' Lemma (a Lagrange multiplier)
λ_D^*	solution of equation (3.13)'
μ, μ_B	constant of Gibbs' Lemma (a Lagrange multiplier)